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Minimal reductions and cores of edge ideals

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ABSTRACT

We study minimal reductions of edge ideals of graphs and determine restrictions on the coefficients of the generators of these minimal reductions. We prove that when I is not basic, then core $I \subset \mathfrak{m}I$, where I is an edge ideal in the corresponding localized polynomial ring and $\mathfrak{m}$ is the maximal ideal of this ring. We show that the inclusion is an equality for the edge ideal of an even cycle with an arbitrary number of whiskers. Moreover, we show that the core is obtained as a finite intersection of homogeneous minimal reductions in the case of even cycles. The formula for the core does not hold in general for the edge ideal of any graph and we provide a counterexample. In particular, we show in this example that the core is not obtained as a finite intersection of general minimal reductions.

1. Introduction

Let R be a Noetherian ring and I an ideal of R. Recall that a *reduction* of I is an ideal J such that $J \subset I$ and $\overline{I} = \overline{J}$, where $\overline{}$ denotes the integral closure. Equivalently, $J \subset I$ is a reduction of I if and only if $I^{r+1} = JI^r$ for some nonnegative integer r [13]. When R is a Noetherian local ring then we may consider minimal reductions, where minimality is with respect to inclusion. Northcott and Rees proved that when R is a Noetherian local ring with infinite residue field then either I has infinitely many minimal reductions or I is *basic*, i.e. I is the only reduction of itself.

A reduction can be thought of as a simplification of the ideal. One advantage to considering reductions is that they are in principle smaller ideals with the same asymptotic behavior as the ideal I itself. For example, all minimal reductions of I have the same height, the same radical, and the same multiplicity as I.

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Let R be a Noetherian local ring with infinite residue field and I an ideal of R. Then every minimal reduction J of I has the same minimal number of generators, $\ell(I)$, where $\ell(I)$ is the *analytic spread* of I (see Section 2). It is well known that every minimal generating set of a reduction J of I can be extended to a minimal generating set of I. Therefore $\ell(I) \leqslant \mu(I)$, where $\mu(I)$ denotes the minimal number of generators of I. When $\ell(I) = \mu(I)$ then I is basic.

Minimal reductions are not unique and therefore one considers the intersection of all the reductions of an ideal, namely the *core* of the ideal. This object was defined by Rees and Sally [16]. When R is a Noetherian local ring it is enough to consider the intersection of the minimal reductions. This intersection is in general infinite and there is significant difficulty in obtaining closed formulas that describe the core. Several authors have determined formulas that compute the core under various assumptions; Corso, Huneke, Hyry, Polini, Smith, Swanson, Trung, Ulrich, Vitulli to name a few, [2,3, 7–10,14,15]. Furthermore, Hyry and Smith have discovered a connection with a celebrated conjecture by Kawamata on the non-vanishing of sections of line bundles [9]. They prove that the validity of the conjecture is equivalent to a statement about *gradedcore*, thus renewing the interest in understanding the core. The gradedcore is the intersection of all homogeneous minimal reductions and in general, $\operatorname{gradedcore}(I) \subset \operatorname{core}(I)$. In Section 5 we provide an instance where equality holds.

In [15] Polini, Ulrich and Vitulli study the core of 0-dimensional monomial ideals in polynomial rings. They prove that the core is obtained by computing the mono of a general locally minimal reduction of I [15, Theorem 3.6]. The mono of an ideal K is the largest monomial subideal contained in K. They provide an effective algorithm for computing the core, which is implemented in computer algebra programs such as CoCoA. In general, though, the question of what is the core of a monomial ideal is quite open.

It was shown in [18, Proposition 2.1] that among the monomial reductions of a monomial ideal, there is a unique minimal element. However, this reduction need not be minimal among all reductions. If a monomial ideal I has a square-free generating set, then Singla showed that the only monomial reduction of I is I itself [18, Remark 2.4]. This leaves a large class of monomial ideals whose minimal reductions are not monomial. Even though a monomial ideal need not have monomial minimal reductions, its core is monomial [2, Remark 5.1].

The class of square-free monomial ideals generated in degree two can be viewed as edge ideals of graphs (see Section 2). Such ideals were introduced in [22] and their properties have been studied by many authors, including [1,4,5,11,12,17,23]. In order to discuss minimal reductions, the ring needs to be a local ring with infinite residue field. Since I is a homogeneous ideal, we will view I as an ideal in the localization of a polynomial ring at its homogeneous maximal ideal $\mathfrak{m}$ and we will assume that the residue field is infinite. By abuse of notation we will still denote the ideal by $I = I(G)$, where G is the associated graph. We note here that the edge ideals we study are far from being 0-dimensional, so the monomial ideals we consider are not in the same class as the ones considered by Polini, Ulrich, and Vitulli in [15].

As mentioned earlier, $\ell(I) \leqslant \mu(I)$ and when $\ell(I) = \mu(I)$ then the ideal is basic. In this case the core is trivial, i.e. $\operatorname{core}(I) = I$. When I is an ideal with $\ell(I) = \mu(I) - 1$ then I is called an ideal of *second analytic deviation one*. For these ideals we show that if $(h_1, \ldots, h_s)$ is a minimal generating set of I, then J has a generating set of the form $(h_1 + a_1h_t, h_2 + a_2h_t, \ldots, h_s + a_sh_t)$ for some $1 \leqslant t \leqslant s$, where $a_i \in R$ for all i and $a_t = -1$ (Lemma 3.2). In Corollary 3.3 we extend this to give a description of the structure of minimal reductions of any ideal in a Noetherian local ring. Not all choices of a_i will result in a reduction, even when the second analytic deviation is one. One of the goals of this paper is to find restrictions on the coefficients a_i. When I is the edge ideal of a graph with a unique even cycle of length d then I is an ideal of second analytic deviation one (Remark 2.1). We show that if $\prod_{i=1}^{\frac{d}{2}} a_{2i-1} = \prod_{j=1}^{\frac{d}{2}} a_{2j}$ then J is not a reduction of I (Corollary 3.8). The condition that J is a minimal reduction of I is an open condition, i.e. the vectors of the coefficients a_i are in a dense open subset of $\mathbb{A}_R^{s-1}$. More precisely, we show that there exists a hypersurface defined by the relation on the products of the coefficients a_i as above, in the complement of this open set.

Let I be the edge ideal of a graph that is not basic and let R be the corresponding localized polynomial ring. Let $\mathfrak{m}$ be the maximal ideal of R. We show in Theorem 4.1 that $\operatorname{core}(I) \subset \mathfrak{m}I$. To establish a case where equality occurs, we consider the class of edge ideals of even cycles with an arbitrary number of whiskers (potentially none) at each vertex. Let I be such an ideal. We show that $J : I = \mathfrak{m}$ for all minimal reductions J of I, Theorem 4.4. In particular, these results imply that $J : I$ is independent of the choice of the minimal reduction J of I. This means that I is a balanced ideal in the sense of [20]. This balanced property allows us to compute a formula for the core of these ideals.

Let R be a Gorenstein local ring and let I be an ideal of R that satisfies G_ℓ and is weakly $(\ell - 1)$-residually S_2, where $\ell = \ell(I)$. Under these assumptions Corso, Polini and Ulrich prove that $\operatorname{core}(I) = (J : I)J = (J : I)I$ for any minimal reduction J of I [3, Theorem 2.6]. The edge ideals we consider are not weakly $(\ell - 1)$-residually S_2. Nonetheless, we establish the same formula for the core for a new class of ideals, namely for the edge ideals described above, Theorem 4.6.

The contents of this paper are as follows. We provide necessary definitions and background material in Section 2. In Section 3 we discuss the format of minimal reductions and restrictions on the coefficients of their generators. In Section 4 we prove the main results of the paper, namely that if I is the edge ideal of any graph, then either I is basic or $\operatorname{core}(I) \subset \mathfrak{m}I$, Theorem 4.1, and if I is the edge ideal of an even cycle with an arbitrary number of whiskers then $J : I = \mathfrak{m}$ for every minimal reduction J of I, Theorem 4.4, and $\operatorname{core}(I) = \mathfrak{m}I$, Theorem 4.6. We give an example of a graph that is neither basic nor a whiskered even cycle for which this formula for the core does not hold, Example 4.8, and the core is not a finite intersection of general minimal reductions. Furthermore, Example 4.8 establishes that the condition that I is weakly $(\ell - 1)$-residually S_2 in [2, Theorem 4.5] is necessary.

In general, the edge ideals of even cycles need not be weakly $(\ell - 1)$-residually S_2. Therefore $\operatorname{core}(I)$ is not a priori a finite intersection of general minimal reductions in this case. Nevertheless, in Section 5 we show that the core of an even cycle is obtained via a finite intersection of homogeneous binomial minimal reductions. It turns out these minimal binomial reductions also establish the gradedcore. We show that $\operatorname{gradedcore}(I) = \operatorname{core}(I)$ for the edge ideals of even cycles, Remark 5.7.

2. Background

Let R be a Noetherian ring and I an ideal. Suppose that $I = (h_1, \ldots, h_q)$. The *Rees algebra* of I is the subring $\mathcal{R}(I) = R[It] = R \oplus It \oplus I^2t^2 \oplus \cdots \subset R[t]$. There is a canonical epimorphism $\phi : A = R[T_1, \ldots, T_q] \longrightarrow \mathcal{R}(I)$ given by $T_i \mapsto h_i t$. Let $L = \ker(\phi)$. Then $L = \bigoplus_{i=1}^{\infty} L_i$ is a graded ideal. The ideal I is said to be of *linear type* if $L = L_1 A$. It follows that $J \subset I$ is a reduction of I if and only if $\mathcal{R}(I)$ is integral over $\mathcal{R}(J)$. Note that if I is an ideal of linear type then I is basic.

Suppose $(R, \mathfrak{m}, k)$ is a Noetherian local ring with infinite residue field and I is an ideal of R. The *special fiber ring* of I is the graded algebra $\mathcal{F}(I) = \mathcal{R}(I) \otimes k = \bigoplus_{i \geqslant 0} I^i/\mathfrak{m}I^i$. As above there is a canonical epimorphism $\psi : B = k[T_1, \ldots, T_q] \longrightarrow \mathcal{F}(I)$, whose kernel is a graded ideal referred to as the ideal of equations of $\mathcal{F}(I)$.

Northcott and Rees proved that when R is a Noetherian local ring then the minimal reductions correspond to Noether normalizations of $\mathcal{F}(I)$ [13]. Furthermore, all minimal reductions have the same minimal number of generators. This number is called the *analytic spread* of I and is defined by $\ell(I) = \dim \mathcal{F}(I)$. It then follows that $\mu(J) = \ell(I)$ for every minimal reduction J of I [13]. Throughout let $\ell = \ell(I)$ denote the analytic spread of I.

Explicit descriptions of the Rees algebra, $\mathcal{R}(I)$, and the special fiber ring, $\mathcal{F}(I)$, of an edge ideal I were obtained by Villarreal in [23]. Let G be a graph on a set of vertices $V = \{x_1, \ldots, x_n\}$. Define I to be the ideal generated by all elements of the form $x_i x_j$, where $\{x_i, x_j\}$ is an edge of G. Then $I = I(G)$ is the *edge ideal* associated to the graph G. In general, I is an ideal of the polynomial ring $k[x_1, \ldots, x_n]$ over a field k. As mentioned in Section 1, in order to discuss minimal reductions of edge ideals of graphs, we will view I as an ideal of the local ring $R = k[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}$, where k is an infinite field.

Villarreal characterized the edge ideals that are of linear type. More precisely, he showed that if G is a connected graph then the edge ideal of G is of linear type if and only if G is a tree or has a unique cycle of odd length [23, Corollary 3.2]. Since the edge ideals of odd cycles or trees are of linear type and hence have no proper reductions, these are precisely the graphs whose edge ideal is basic. Thus we will consider edge ideals of graphs with *irreducible even closed walks*. Here a closed walk $x_1, e_1, x_2, e_2, x_3, \ldots, e_d, x_1$ is considered to be reducible if there exist edges e_i and e_j in the walk such that $e_i = e_j$ and i and j have different parities. Such walks are considered reducible because they do not correspond to minimal relations of the defining ideal of the fiber cone [23, Proposition 3.1]. Note that a graph G contains an irreducible even closed walk if and only if G is not of linear type. Just as for a cycle, a closed walk is considered to be independent of its starting point for the purpose of uniqueness. This also allows an even closed walk to be represented by its edges with the vertices suppressed. Note that if $e_1, \dots, e_d$ is an even closed walk, then $e_1, \dots, e_d, e_1, \dots, e_d$ is an even closed walk, which will be considered as a multiple of $e_1, \dots, e_d$. A graph will be considered to have a unique irreducible even closed walk if all irreducible even closed walks are multiples of a fixed irreducible even closed walk.

Even cycles provide examples of irreducible even closed walks. For a more general example of an even closed walk, consider the graph whose edges are $e_1 = x_1x_2$, $e_2 = x_2x_3$, $e_3 = x_1x_3$, $e_4 = x_1x_4$, $e_5 = x_4x_5$, $e_6 = x_1x_5$. Then $e_1, e_2, e_3, e_4, e_5, e_6$ is an irreducible even closed walk without repeated edges that has a repeated vertex. For a nontrivial example of an irreducible even closed walk with repeated edges, consider the walk $e_1, e_2, e_3, e_4, e_5, e_6, e_3, e_7$ in the graph whose edges are $e_1 = x_1x_2$, $e_2 = x_2x_3$, $e_3 = x_3x_4$, $e_4 = x_4x_5$, $e_5 = x_5x_6$, $e_6 = x_6x_4$, $e_7 = x_3x_1$. Notice that if we label the edges of the walk $f_1, \dots, f_8$, then $f_3 = f_7$ and 3, 7 have the same parity.

Remark 2.1. Let G be a graph with s edges and a unique irreducible even closed walk given by $e_{i_1}, e_{i_2}, \dots, e_{i_d}$, and let $I = I(G)$ be the edge ideal of G. Then $\mathcal{F}(I) \simeq k[T_1, T_2, \dots, T_s]/(T_{i_1}T_{i_3}\cdots T_{i_{d-1}} - T_{i_2}T_{i_4}\cdots T_{i_d})$, by [23, Proposition 3.1]. Therefore $\ell = s-1$ and I is an ideal of second analytic deviation one.

3. The structure of minimal reductions

We begin by proving a general result about the form of a minimal reduction of an ideal I of second analytic deviation one. We state the following lemma for ease of reference.

Lemma 3.1. *(See [13].) Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let I, K be ideals such that $K \subset I$ and $\overline{K + \mathfrak{m}I} = \bar{I}$, where $\bar{I}$ denotes the integral closure of I. Then $\overline{K} = \bar{I}$, i.e. K is a reduction of I.*

Lemma 3.2. *Let R be a Noetherian local ring with infinite residue field. Assume I is an ideal with $\ell = \mu(I) - 1$, and let J be a minimal reduction of I. If $(h_1, \dots, h_s)$ is a minimal generating set of I, then J has a generating set of the form $(h_1 + a_1h_t, h_2 + a_2h_t, \dots, h_s + a_sh_t)$ for some $1 \leqslant t \leqslant s$, where $a_i \in R$ for all i and $a_t = -1$.*

Proof. Let $I = (h_1, \dots, h_s)$ and let J be a minimal reduction of I. If $s = 1$ then the result is trivial. Suppose that $s \geqslant 2$. Then $J = (f_1, \dots, f_{s-1})$ for some $f_i \in I$. Let $f_i = \sum_{j=1}^{s} a_{ij}h_j$ and let $A = (a_{ij})$ be the matrix of coefficients of J. Then A is an $(s-1) \times s$ matrix. Let $\mathfrak{m}$ denote the unique maximal ideal of R.

Suppose that $a_{ij} \in \mathfrak{m}$ for all i and j. Then $J \subset \mathfrak{m}I \subset I$. As $\bar{J} = \bar{I}$ then $\overline{0 + \mathfrak{m}I} = \bar{I}$. Hence by Lemma 3.1 we have 0 is a reduction of I, which is impossible. Therefore $a_{ij} \notin \mathfrak{m}$ for some a_{ij}. After reordering the h_i and the f_i we may assume, without loss of generality, that $a_{11} = 1$. Using row operations, which correspond to changing the generating set of J, we can assume that A has the form

$$\begin{pmatrix} 1 & a_{12} & \cdots & a_{1,s-1} & a_{1,s} \\ 0 & a_{22} & \cdots & a_{2,s-1} & a_{2,s} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{s-1,2} & \cdots & a_{s-1,s-1} & a_{s-1,s} \end{pmatrix}.$$

Notice that J is minimally generated by $s-1$ elements ([13] or [19, Proposition 8.3.7]). Hence the matrix A has full rank and thus using an argument similar to the one above we may row reduce A and assume that it is of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & a_{1,s} \\ 0 & 1 & \cdots & 0 & a_{2,s} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{s-1,s} \end{pmatrix}.$$

Then we may write J as $J = (h'_1 + a_{1,s}h'_s, \ldots, h'_i + a_{i,s}h'_s, \ldots, h'_{s-1} + a_{s-1,s}h'_s)$, where $a_{i,s} \in R$ and $h'_i = h_{\sigma(i)}$ for some permutation σ of $\{1, \ldots, s\}$. The result follows by setting $t = \sigma(s)$, $a_t = -1$, and $a_{\sigma(i)} = a_{i,s}$ for all $1 \leqslant i \leqslant s-1$. $\square$

The proof of Lemma 3.2 can be extended for ideals with arbitrary second analytic deviation.

Corollary 3.3. *Let R be a Noetherian local ring with infinite residue field. We assume I is an ideal with $\ell = \mu(I) - n = s - n$, and let J be a minimal reduction of I. If $(h_1, \ldots, h_s)$ is a minimal generating set of I, then J has a generating set of the form*

$$(h_1 + a_{1,1}h_{t_1} + \cdots + a_{1,n}h_{t_n}, \ldots, h_s + a_{s,1}h_{t_1} + \cdots + a_{s,n}h_{t_n})$$

for some $1 \leqslant t_1, \ldots, t_n \leqslant s$, where $a_{i,j} \in R$ for all i, j and $a_{t_i,j} = -\delta_{ij}$ for all $1 \leqslant i, j \leqslant n$.

Next we give an interpretation of Corollary 3.3 in the case of an edge ideal that contains a unique irreducible even closed walk.

Corollary 3.4. *Let $I = (e_1, \ldots, e_s)$ be the edge ideal of a graph with s edges containing a unique irreducible even closed walk and let J be a minimal reduction of I. Then J is of the form $(e_1 + a_1e_t, e_2 + a_2e_t, \ldots, e_s + a_se_t)$ for some $1 \leqslant t \leqslant s$, where $a_i \in R$ for all i and $a_t = -1$.*

Proof. This follows immediately from Lemma 3.2 and Remark 2.1. $\square$

In addition to knowing the general form a reduction can take we also have control over the reduction number for the edge ideal of a graph with a unique irreducible even closed walk.

Let R be a Noetherian local ring, I an ideal of R and let J be a minimal reduction of I. The smallest r for which the equality $I^{r+1} = JI^r$ holds is called the *reduction number of I with respect to J* and is denoted by $r_J(I)$. The reduction number $r_J(I)$ provides a measure of how closely related J is to I. The *reduction number*, $r(I)$, of I is the minimum of the reduction numbers $r_J(I)$, where J ranges over all minimal reductions of I.

Lemma 3.5. *Let I be the edge ideal of a graph with s edges containing a unique irreducible even closed walk, which is of length d. Then $r_J(I) = \frac{d}{2} - 1$ for any minimal reduction J of I. In particular, $r_J(I)$ is independent of the minimal reduction J of I.*

Proof. By [23, Proposition 3.1] we know that the special fiber ring of I is

$$\mathcal{F}(I) \simeq k[T_1, T_2, \ldots, T_s]/(T_{i_1}T_{i_3} \cdots T_{i_{d-1}} - T_{i_2}T_{i_4} \cdots T_{i_d}),$$

where $e_{i_1}, \ldots, e_{i_d}$ are the not necessarily distinct edges of the even walk. Since the degree of the relation in the defining ideal of $\mathcal{F}(I)$ is $\frac{d}{2}$ then it follows that $r_J(I) = \frac{d}{2} - 1$ by [21, Proposition 5.1.3]. $\square$

The next lemma and proposition allow us to use counting arguments to eliminate potential reductions.

Lemma 3.6. *Let $I = (e_1, \dots, e_s)$ be the edge ideal of a graph with s edges, and let $J = (e_1 + a_1 e_s, \dots, e_{s-1} + a_{s-1} e_s)$. Fix $r \geqslant 2$ and define K^{r-1} to be the ideal generated by all elements of the form $(e_i + a_i e_s)e_{i_1} \cdots e_{i_{r-1}}$ where $i \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{r-1}$. Then $JI^{r-1} = K^{r-1}$.*

Proof. For clarity, we first handle the case $r = 2$. Clearly $K \subset JI$. Since JI can be generated by elements of the form $(e_q + a_q e_s)e_{i_1}$, we consider a generator $(e_q + a_q e_s)e_{i_1} \in JI$ for some $i_1 < q < s$. Then

$$(e_q + a_q e_s)e_{i_1} = (e_{i_1} + a_{i_1} e_s)e_q - a_{i_1}(e_q + a_q e_s)e_s + a_q(e_{i_1} + a_{i_1} e_s)e_s \in K.$$

Thus $JI = K$.

For the general case, consider a generator $(e_q + a_q e_s)M \in JI^{r-1}$, where M is a monomial generator of I^{r-1}. Write $M = e_{i_1} e_{i_2} \cdots e_{i_{r-1}}$ with $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{r-1}$. Assume $i_1 < q$, and let $N = e_{i_2} e_{i_3} \cdots e_{i_{r-1}}$. Then multiplying the equation above by N yields

$$(e_q + a_q e_s)e_{i_1} N = (e_{i_1} + a_{i_1} e_s)N e_q - a_{i_1}(e_q + a_q e_s)N e_s + a_q(e_{i_1} + a_{i_1} e_s)N e_s.$$

Now by the choice of i_1, $(e_{i_1} + a_{i_1} e_s)N e_q \in K^{r-1}$, as is $(e_{i_1} + a_{i_1} e_s)N e_s$. Consider $(e_q + a_q e_s)N e_s$. If $i_2 \geqslant q$ we are done. Otherwise, repeat the process for $N e_s$. Since M is a product of $r - 1$ edges, this process must terminate. Thus $JI^{r-1} \subset K^{r-1}$. Since the other inclusion is clear, $JI^{r-1} = K^{r-1}$ as claimed. $\square$

Proposition 3.7. *Let $I = (e_1, \dots, e_s)$ be the edge ideal of a graph with s edges containing a unique irreducible even closed walk, which is of length d. Let $J = (e_1 + a_1 e_t, \dots, e_s + a_s e_t)$ for some $1 \leqslant t \leqslant s$, where $a_i \in R$ and $a_t = -1$. Then*

$$\mu(I^r) = \begin{cases} \binom{s+r-1}{r}, & r < \frac{d}{2}, \\ \binom{s+r-1}{r} - 1, & r = \frac{d}{2} \end{cases}$$

and $\mu(JI^{r-1}) \leqslant \binom{s+r-1}{r} - 1$ for $r \geqslant 1$.

Proof. The number of products, allowing for repetition, of r elements selected from a set containing s elements is $\binom{s+r-1}{r}$, so I^r can be generated by $\binom{s+r-1}{r}$ monomials. From the structure of the fiber ring of I, Remark 2.1, we know that there are no relations among the generators in degree less than $\frac{d}{2}$, and there is precisely one relation in degree $\frac{d}{2}$. Thus if $r < \frac{d}{2}$, there are no relations among the products counted and the result follows. If $r = \frac{d}{2}$ and the edges of the irreducible even closed walk are $e_{i_1}, \dots, e_{i_d}$, then $e_{i_1} e_{i_3} \cdots e_{i_{d-1}} = e_{i_2} e_{i_4} \cdots e_{i_d}$ has been counted twice. Note that there are no other relations in degree $\frac{d}{2}$ and thus $\mu(I^r) = \binom{d+r-1}{r} - 1$ for $r = \frac{d}{2}$.

Assume J is an ideal of the given form. Select any relabeling of the edges of G so that $t = s$. By Lemma 3.6, in order to provide an upper bound on the minimal number of generators of JI^{r-1}, it suffices to provide an upper bound on the minimal number of generators of K^{r-1}. Note that for any $1 \leqslant i < s$, there are $s - i + 1$ generators of I from which $r - 1$ are selected, with possible repetition, to form a monomial M for which $(e_i + a_i e_s)M$ is a generator of K^{r-1}. There are $\binom{s-i+1+r-1-1}{r-1}$ possible generators of K^{r-1} of the form $(e_i + a_i e_s)M$ for each $1 \leqslant i < s$. Now we have that $\sum_{i=1}^{s} \binom{s+r-1-i}{r-1} = \binom{s+r-1}{r}$. Thus there are

$$\sum_{i=1}^{s-1} \binom{s+r-1-i}{r-1} = \binom{s+r-1}{r} - \binom{s+r-1-s}{r-1} = \binom{s+r-1}{r} - 1$$

elements in the generating set described above for $K^{r-1} = JI^{r-1}$. This gives the desired upper bound on $\mu(JI^{r-1})$. $\square$

Note that when $r < \frac{d}{2}$ the bound given above on the number of generators of JI^{r-1} is actually an equality. To see this, write $J = (f_1, \dots, f_{s-1})$ and $I = (J, f_s)$ for some choice of f_i. Then among the generators $f_{i_1} \cdots f_{i_r}$ of I^r, the only one that is not automatically in JI^{r-1} is f_s^r. Since Proposition 3.7 shows that I^r has $\binom{s+r-1}{r}$ distinct generators for $r < \frac{d}{2}$, this gives at least $\binom{s+r-1}{r} - 1$ distinct generators of JI^{r-1}. Thus if $r < \frac{d}{2}$ then $\mu(JI^{r-1}) = \binom{s+r-1}{r} - 1$.

Using the information about the reduction numbers from Lemma 3.5 we show that the counting arguments used in Proposition 3.7 impose restrictions on the coefficients of the generators of the reductions in the case of edge ideals of graphs with a unique even cycle. Note that the proof below easily generalizes to graphs containing a unique even closed walk that does not contain repeated edges. Throughout the remainder of the paper, it will be convenient to reorder the edges of a cycle so that a particular edge is last. To that end, assume $e_1, \dots, e_d$ form an even cycle, where $e_i = x_i x_{i+1}$ for $1 \leqslant i < d$ and $e_d = x_1 x_d$. We define a *cyclic reordering* of the vertices to be a relabeling σ of the vertices such that $\sigma(x_i) = x_{i+j}$ for some fixed j, where subscripts are taken modulo d and $\bar{0} = d$. Such a reordering preserves adjacencies and the cycle structure, but allows any particular edge of the cycle to be considered last, namely as e_d.

Corollary 3.8. *Let $I = (e_1, \dots, e_s)$ be the edge ideal of a graph with s edges containing a unique even cycle, $e_1, \dots, e_d$. Define $J = (e_1 + a_1 e_t, \dots, e_s + a_s e_t)$ for some $1 \leqslant t \leqslant s$, where $a_i \in R$ and $a_t = -1$. If $\prod_{i=1}^{\frac{d}{2}} a_{2i-1} = \prod_{j=1}^{\frac{d}{2}} a_{2j}$, then J is not a reduction of I.*

Proof. If J is a reduction of I, then J must be minimal since it has ℓ generators. By Lemma 3.5, we know that J is a minimal reduction of I if and only if $JI^{r-1} = I^r$, where $r = \frac{d}{2}$.

There are two cases to consider. If $t \leqslant d$, then after a cyclic reordering of the cycle we may assume $t = d$ and $a_d = -1$. Otherwise, $t > d$. Assume $\prod_{i=1}^{\frac{d}{2}} a_{2i-1} = \prod_{j=1}^{\frac{d}{2}} a_{2j}$. Using this equality and the relation among the edges of the cycle, it is easy to check that for $t \geqslant d$

$$(e_1 + a_1 e_t) e_3 e_5 \cdots e_{d-1} = \sum_{i=1}^{r} (-1)^{i-1} a_2 a_4 \cdots a_{2i-2} (e_{2i} + a_{2i} e_t) e_t^{i-1} e_d e_{d-2} \cdots e_{2i+2}$$

$$+ \sum_{i=1}^{r-1} (-1)^{i-1} a_1 a_3 \cdots a_{2i-1} (e_{2i+1} + a_{2i+1} e_t) e_t^{i} e_{d-1} \cdots e_{2i+3},$$

where empty products are defined to be one. Note that this is a relation among the generators of K^{r-1} that were counted in Proposition 3.7. Therefore by Lemma 3.6, $\mu(JI^{r-1}) = \mu(K^{r-1}) \leqslant \binom{d+r-1}{r} - 1 - 1 < \mu(I^r)$. Thus J is not a reduction of I. $\square$

We conclude this section by providing concrete examples of reductions for the edge ideals of graphs containing a unique irreducible even closed walk. Note that these examples will provide the building blocks for computing the core as a finite intersection in Section 5.

Example 3.9. Let I be the edge ideal of a graph of an even cycle. Let R be the corresponding localized polynomial ring and let k be the residue field of R. We further assume that the characteristic of k is not 2. Let $J = (e_1 + a_1 e_t, \dots, e_d + a_d e_t)$ for some $1 \leqslant t \leqslant d$, where $a_i = 1$ for all $i \neq t$ and $a_t = -1$. Then J is a minimal reduction of I.

Proof. If J is a reduction of I, then J is a minimal reduction since $J \subset I$ and $\mu(J) = \ell$. After a cyclic reordering we may assume $t = d$ and $a_d = -1$. Let $r = \frac{d}{2}$. Clearly $JI^{r-1} \subset I^r$. To see the other inclusion, we first prove $e_d^r \in JI^{r-1}$. Notice that $e_d^r + (-1)^{r-1}\prod_{i=1}^r e_{2i-1} \in JI^{r-1}$ since

$$e_d^r + (-1)^{r-1}\prod_{i=1}^{r} e_{2i-1} = \sum_{i=1}^{r}(-1)^{i-1}(e_{2i-1} + e_d)e_1 \cdots e_{2i-3}e_d^{r-i},$$

where empty products are defined to be one. Similarly, $e_d^r + (-1)^{r-2}\prod_{j=1}^r e_{2j} \in JI^{r-1}$ since $e_d^r + (-1)^{r-2}\prod_{j=1}^r e_{2j} = \sum_{i=1}^{r-1}(-1)^{i-1}(e_{2i} + e_d)e_2 \cdots e_{2i-2}e_d^{r-i}$. Combining these relations with the relation on the edges $\prod_{i=1}^r e_{2i-1} = \prod_{j=1}^r e_{2j}$ gives $2e_d^r \in JI^{r-1}$. Thus $e_d^r \in JI^{r-1}$ as desired.

Now let $M \in I^r$ be a monomial generator. If $M = e_d^r$ we are done by the argument above. If not, write $M = e_{i_1}e_{i_2}\cdots e_{i_r}$ for some choice of r edges, ordered so that $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r$. Define $M_1 = e_{i_2}e_{i_3}\cdots e_{i_r}$ and consider $(e_{i_1} + e_d)M_1 = M + e_dM_1$. If $M_1 = e_d^{r-1}$, then since e_dM_1 and $(e_{i_1} + e_d)M_1$ are both in JI^{r-1}, we see that $M \in JI^{r-1}$ as well. If $M_1 \neq e_d^{r-1}$, then define $M_2 = e_{i_3}e_{i_4}\cdots e_{i_r}$. Notice that if $M_2 = e_d^{r-2}$, then by the equation $(e_{i_2} + e_d)M_2 = M_1 + e_dM_2$ one sees that $M_1 \in JI^{r-2}$ as above, which then implies $M \in JI^{r-1}$. If $M_2 \neq e_d^{r-2}$ we repeat the process. The process is clearly finite, and since at each stage of the algorithm, M_i is replaced by e_dM_{i+1}, the algorithm will terminate. Thus for some (not necessarily distinct) edges e_{i_j}, $M + (-1)^{q-1}e_d^r = (e_{i_1} + e_d)M_1 - (e_{i_2} + e_d)e_dM_2 + \cdots + (-1)^{q-1}(e_{i_q} + e_d)e_d^{q-1}M_q$, where $q \leqslant \frac{d}{2}$ and $M_q = e_d^{r-q}$. Thus $M \in JI^{r-1}$. $\square$

Example 3.9 generalizes to even closed walks without repeated edges. We remark that when $\operatorname{char} k = 2$ then it follows immediately from Corollary 3.8 that the ideal J in Example 3.9 is not a minimal reduction of I. In order to avoid characteristic dependent arguments, we provide two additional examples of minimal reductions that are free of characteristic assumptions and which hold for edge ideals of graphs containing a (not necessarily unique) irreducible even closed walk.

Example 3.10. Let I be the edge ideal of a graph containing an irreducible even closed walk $e_1, \ldots, e_d$. Write $I = (e_1, \ldots, e_d, e_{d+1}, \ldots, e_s)$, where $e_{d+1}, \ldots, e_s$ are the distinct edges of G not contained in the walk. Define $\delta'_{i,j}$ to be -1 if $e_i = e_j$ and 1 otherwise. Then $J = (e_1, e_2 + \delta'_{2,d}e_d, e_3, \ldots, e_{d-2} + \delta'_{d-2,d}e_d, e_{d-1}, e_{d+1} + e_d, \ldots, e_s + e_d)$ is a reduction of I. Furthermore, if I contains a unique irreducible even closed walk, then J is a minimal reduction of I.

Proof. Note that the first d generators of I are not necessarily unique, but that any repeated edges will have the same parity. Also, any repeated edge other than e_d listed in the generating set of I corresponds to a repeated generator of J. Hence $\mu(J) = \mu(I) - 1$. Let $r = \frac{d}{2}$. Clearly $JI^{r-1} \subset I^r$. For the other inclusion, let M be a monomial generator of I^r. Write $M = e_{i_1}e_{i_2}\cdots e_{i_r}$ for some choice of r edges, where if a repeated edge divides M, the largest possible subscript for the edge is used. If i_j is odd and less than d for some j, then $M = e_{i_j}N$, where $e_{i_j} \in J$ and $N \in I^{r-1}$. Thus $M \in JI^{r-1}$. So suppose i_j is not odd for all $i_j < d$. Define s_i to be the number of times that $e_j = e_d$ for $j < i$. As in Example 3.9 we have

$$e_d^r + (-1)^{r-2-s_d}\prod_{j=1}^{r} e_{2j} = \sum_{i=1}^{r-1}(-1)^{i-1-s_{2i}}\left(e_{2i} + \delta'_{2i,d}e_d\right)e_2 \cdots e_{2i-2}e_d^{r-i} \in JI^{r-1}.$$

By the relation $\prod_{i=1}^r e_{2i-1} = \prod_{j=1}^r e_{2j}$ and the fact that $\prod_{i=1}^r e_{2i-1} \in JI^{r-1}$ we have that $e_d^r \in JI^{r-1}$. The remainder of the argument follows as in Example 3.9 by noting that each e_{i_j} in the expression for M now has i_j even or $i_j \geqslant d$ and thus $(e_{i_j} + e_d)M_j \in JI^{r-1}$ for each j. Finally, when I contains a unique irreducible even closed walk then $\ell = \mu(I) - 1$. Hence J is a minimal reduction of I. $\square$

Example 3.11. Let I be the edge ideal of a graph containing an irreducible even closed walk $e_1, \dots, e_d$. Write $I = (e_1, \dots, e_d, e_{d+1}, \dots, e_s)$, where $e_{d+1}, \dots, e_s$ are the distinct edges of G not contained in the walk. Define $\delta_{i,j}$ to be 0 if $e_i = e_j$ and 1 otherwise. Then $J = (e_1 + e_d, \delta_{2,d}e_2, e_3 + e_d, \dots, \delta_{d-2,d}e_{d-2}, e_{d-1} + e_d, e_{d+1} + e_d, \dots, e_s + e_d)$ is a reduction of I. Furthermore, if I contains a unique irreducible even closed walk, then J is a minimal reduction of I.

Proof. The proof is similar to the proof of Example 3.10. $\square$

4. Cores of edge ideals of whiskered cycles

Recall that if I is the edge ideal of a connected graph, then I is of linear type if and only if I is the edge ideal of a tree or of a graph containing a unique cycle of odd length by [23, Corollary 3.2], and thus $\operatorname{core}(I) = I$. This implies that I is not of linear type if and only if the graph associated to I has an irreducible even closed walk. In this section, we show that if I is the edge ideal of any graph that is not basic, then we have $\operatorname{core}(I) \subset \mathfrak{m}I$. We also establish a class of graphs for which this inclusion is an equality. Note that the core of a monomial ideal is also a monomial ideal by [2, Remark 5.1].

Theorem 4.1. *Let I be the edge ideal of a connected graph containing an irreducible even closed walk. Then* $\operatorname{core}(I) \subset \mathfrak{m}I$.

Proof. Write $I = (e_1, \dots, e_s)$, where $e_1, \dots, e_d$ form an irreducible even closed walk. Let e_i be a generator of I. If i is odd then

$$J_1 = (e_1 + e_d, \delta_{2,d}e_2, \dots, \delta_{d-2,d}e_{d-2}, e_{d-1} + e_d, e_{d+1} + e_d, \dots, e_s + e_d)$$

is a reduction of I by Example 3.11 and $e_i \notin J_1$. Similarly, if i is even then $J_2 = (e_1, e_2 + \delta'_{2,d}e_d, \dots, e_{d-2} + \delta'_{d-2,d}e_d, e_{d-1}, e_{d+1} + e_d, \dots, e_s + e_d)$ is a reduction of I by Example 3.10 and $e_i \notin J_2$. Therefore $e_i \notin \operatorname{core}(I)$.

Let g be a minimal monomial generator of $\operatorname{core}(I)$. Since $g \in I$ then $g = fe_i$ for some e_i and $f \in R$ a monomial. Since $e_i \notin \operatorname{core}(I)$ then $f \in \mathfrak{m}$. Therefore $g \in \mathfrak{m}I$ and thus $\operatorname{core}(I) \subset \mathfrak{m}I$. $\square$

We state the following result without a proof, as its proof is elementary.

Lemma 4.2. *Let R be a commutative ring with identity, let $d \geqslant 4$ be an even integer, and let $b_1, \dots, b_d \in R$. Let B be a $d \times d$ matrix of the following form*:

$$B = \begin{pmatrix} 0 & b_d & 0 & 0 & \dots & 0 & -b_1 \\ -b_2 & 0 & b_1 & 0 & 0 & \dots & 0 \\ 0 & -b_3 & 0 & b_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{d-1} & 0 & 0 & \dots & 0 & -b_d & 0 \end{pmatrix}.$$

Then $\det B = (\prod_{i=1}^{\frac{d}{2}} b_{2i-1} - \prod_{j=1}^{\frac{d}{2}} b_{2j})^2$.

For the rest of the article we will assume that I is the edge ideal of a graph G with a unique even cycle and will order the edges so that $e_1, \dots, e_s$ are the edges of G and $e_1, \dots, e_d$ are the edges of the even cycle. In general, if G is a connected graph on n vertices with s edges, then $s \geqslant n$ with equality if and only if G has a unique cycle. Thus for the remainder of the article, the number of edges will be the same as the number of vertices of the graph. For the next theorem, we need to further restrict the class of graphs considered.

Assumptions and Discussion 4.3. Let G be a connected graph on the vertices $x_1, \dots, x_s$ containing a unique cycle, which is of even length $d \geqslant 4$, given by $e_i = x_i x_{i+1}$ for $1 \leqslant i < d$ and $e_d = x_1 x_d$. Assume further that x_j is a leaf for all $j > d$. Thus for each $j > d$ there exists a unique vertex x_{i_j} with $1 \leqslant i_j \leqslant d$ such that $e_j = x_{i_j} x_j$ is an edge of G. Notice that it is not required that the i_j be distinct for different j. It is possible for a single vertex of the cycle to have multiple leaves as neighbors. Let $I = (e_1, \dots, e_s)$ be the edge ideal of G in the localized polynomial ring $R = k[x_1, \dots, x_s]_{(x_1, \dots, x_s)}$ over an infinite field k. Then $\mu(I) = s$, and $\ell = s - 1$ by [23, Proposition 3.1]. We remark that Corollary 3.8 holds for this class of ideals.

The following theorem shows that for the class of edge ideals I with a unique even cycle and an arbitrary number of whiskers, the ideal $J : I$ is independent of the minimal reduction J of I.

Theorem 4.4. *Let R and I be as in* 4.3 *and let J be a minimal reduction of I. Then $J : I = \mathfrak{m}$.*

Proof. Let J be a minimal reduction of I. Then J is of the form $(e_1 + a_1 e_t, \dots, e_s + a_s e_t)$ for some $1 \leqslant t \leqslant s$, where $a_i \in R$ for all i and $a_t = -1$, by Corollary 3.4. Let $f_i = e_i + b_i e_t$, where $b_i = a_i$ if $a_i \notin \mathfrak{m}$ and $b_i = 0$ if $a_i \in \mathfrak{m}$. Consider $J' = (f_1, \dots, f_s)$, where $f_t = 0$ since $b_t = -1$. Notice that $J \subset J' + \mathfrak{m}I \subset I$. Hence J' is a reduction of I by Lemma 3.1.

Consider a presentation matrix ϕ of I, where $R^q \xrightarrow{\phi} R^s \longrightarrow I \longrightarrow 0$. Let ψ be the submatrix of ϕ consisting of the linear relations on the generators of I. Then ψ is an $s \times (2s - d)$ matrix of the form $\psi = (\,\psi_1 \quad \psi_2 \quad \psi_3\,)$, where ψ_1, ψ_2, ψ_3 are matrices defined below. For the remainder of the proof we let $\bar{i} = i$ modulo d, with the convention that $\bar{0} = d$.

Let ψ_1 be an $s \times d$ matrix such that for each $1 \leqslant i \leqslant d$ the i-th column is $(0, \dots, 0, -x_{\overline{i+1}}, x_{\overline{i-1}}, 0, \dots, 0)^T$, where $-x_{\overline{i+1}}$ is the $\overline{(i-1)}$ entry and $x_{\overline{i-1}}$ is the i-th entry.

Let ψ_2 be an $s \times (s - d)$ matrix such that for each $d + 1 \leqslant j \leqslant s$ the $(j - d)$-th column is $(0, \dots, 0, x_j, 0, \dots, 0, -x_{\overline{i_j - 1}}, 0, \dots, 0)^T$, where x_j is the $\overline{(i_j - 1)}$ entry and $-x_{\overline{i_j-1}}$ is the j-th entry.

Let ψ_3 be an $s \times (s - d)$ matrix such that for each $d + 1 \leqslant j \leqslant s$ the $(j - d)$-th column is $(0, \dots, 0, x_j, 0, \dots, 0, -x_{\overline{i_j+1}}, 0, \dots, 0)^T$, where x_j is the i_j entry and $-x_{\overline{i_j+1}}$ is the j-th entry.

We remark that if $s = d$, then the matrices ψ_2 and ψ_3 are zero and the matrix ψ is a $d \times d$ matrix. Notice that performing a series of elementary row operations on ϕ corresponds to altering the generating set of I. We choose elementary row operations so that the generating set of I becomes $I = (J', e_t)$. Let ϕ' be the corresponding presentation matrix of I and ψ' the submatrix consisting of the columns containing the linear relations. By the choice of the generating set, the t-th row of ϕ' forms a (not necessarily minimal) presentation matrix $\tilde{\phi}$ of I/J'. Let $\tilde{\psi}$ denote the t-th row of ψ'. We will show that $I_1(\tilde{\psi}) = I_1(\tilde{\phi}) = \mathfrak{m}$. Notice that

$$\begin{aligned} I_1(\tilde{\psi}) = \big(&\{b_{\overline{i-1}} x_{\overline{i+1}} - b_i x_{\overline{i-1}} \mid \text{for } 1 \leqslant i \leqslant d\}, \\ &\{b_j x_{\overline{i_j-1}} - b_{\overline{i_j-1}} x_j \mid \text{for } d+1 \leqslant j \leqslant s\}, \\ &\{b_j x_{\overline{i_j+1}} - b_{i_j} x_j \mid \text{for } d+1 \leqslant j \leqslant s\}\big). \end{aligned}$$

Then $\tilde{\psi}^T = B \cdot (\underline{x})^T$, where $(\underline{x})^T = (x_1, \dots, x_s)^T$ and $B = \begin{pmatrix} B_1 \\ C \end{pmatrix}$, where $B_1 = (\,B_0 \quad \underline{0}\,)$,

$$B_0 = \begin{pmatrix} 0 & b_d & 0 & 0 & \dots & 0 & -b_1 \\ -b_2 & 0 & b_1 & 0 & 0 & \dots & 0 \\ 0 & -b_3 & 0 & b_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{d-1} & 0 & 0 & \dots & 0 & -b_d & 0 \end{pmatrix},$$

$\underline{0}$ is the $d \times (s-d)$ zero matrix, and C is a $2(s-d) \times s$ matrix. We construct C as follows. For each $d+1 \leqslant j \leqslant s$ there are two rows of C:

$C_j^1 = (0, \dots, 0, b_j, 0, \dots, 0, -b_{\overline{i_j-1}}, 0, \dots, 0)$, where b_j is the $\overline{(i_j-1)}$-th entry and $-b_{\overline{i_j-1}}$ is the j-th entry and

$C_j^2 = (0, \dots, 0, b_j, 0, \dots, 0, -b_{i_j}, 0, \dots, 0)$, where b_j is the $\overline{(i_j+1)}$-th entry and $-b_{i_j}$ is the j-th entry.

Notice that when $s = d$ then $B = B_0$ and $\det B = \det B_0 \neq 0$, by Lemma 4.2 and Corollary 3.8. In general, we will construct an $s \times s$ submatrix of B with a nonzero determinant and thus after row reducing B we will have $I_1(\tilde{\psi}) = \mathfrak{m}$.

We remark that by construction of the submatrix C, for each $d+1 \leqslant j \leqslant s$ the rows C_j^1 and C_j^2 have nonzero entries in the j-th column, one of those entries is $-b_{\overline{i_j-1}}$ and the other is $-b_{i_j}$. Notice that one of $\overline{i_j-1}$ and i_j will be even and one will be odd.

First consider the submatrix $B_1' = \begin{pmatrix} B_1 \\ C_1 \end{pmatrix}$, where C_1 is the submatrix of C constructed by selecting all the rows of C such that for each $d+1 \leqslant j \leqslant s$ the entry in the j-th column is $-b_r$ for some r even. Notice that B_1' is a block matrix and after exchanging rows of C_1 we have a diagonal matrix of size $(s-d) \times (s-d)$ in the lower right corner. Thus after these row operations B_1' is equivalent to $\begin{pmatrix} B_0 & \underline{0} \\ C_1' & D_1 \end{pmatrix}$, where D_1 is diagonal with diagonal entries of the form $-b_r$ with $2 \leqslant r \leqslant d$ even.

Therefore, $\det B_1' = \pm \det B_0 \det D_1$. Since D_1 is diagonal, $\det D_1$ is the product of its diagonal entries. Notice that each diagonal entry of D_1 is by definition of the form b_r for some even $2 \leqslant r \leqslant d$, but not all even r need occur, and some could occur multiple times.

We now consider another $s \times s$ submatrix of B, namely $B_2' = \begin{pmatrix} B_1 \\ C_2 \end{pmatrix}$, where C_2 is the submatrix of C constructed by selecting all the rows of C such that for each $d+1 \leqslant j \leqslant s$ the entry in the j-th column is $-b_q$ such that q is odd. Notice that B_2' is a block matrix and after exchanging rows of C_2 we have a diagonal matrix of size $(s-d) \times (s-d)$ in the lower right corner. Thus B_2' is equivalent to $\begin{pmatrix} B_0 & \underline{0} \\ C_2' & D_2 \end{pmatrix}$, where D_2 is diagonal with diagonal entries of the form $-b_q$ with $1 \leqslant q \leqslant d$ odd. Notice that the diagonal entries of D_2 are not necessarily distinct. As before $\det B_2' = \pm \det B_0 \det D_2$ and $\det D_2$ is a product of its diagonal entries, each of which has an odd subscript.

We observe that $\det B_1'$ and $\det B_2'$ are not simultaneously zero. By Corollary 3.8 and Lemma 4.2 we have $\det B_0 = (\prod_{i=1}^{\frac{d}{2}} b_{2i-1} - \prod_{j=1}^{\frac{d}{2}} b_{2j})^2 \neq 0$. It follows that since each $b_i \in k$ it is not possible to have $b_q = 0$ for some odd q and $b_r = 0$ for some even r simultaneously. Thus $\det D_1$ and $\det D_2$ cannot be simultaneously zero.

Therefore $I_1(\tilde{\psi}) = \mathfrak{m}$. Notice that we have $R^q \xrightarrow{\tilde{\phi}} R \longrightarrow I/J' \longrightarrow 0$ and $I_1(\tilde{\psi}) \subset I_1(\tilde{\phi}) \subset \operatorname{ann}(I/J') = J' : I$. Furthermore, since J' is a minimal reduction of I then $J' : I \neq R$. Hence $I_1(\tilde{\phi}) = \mathfrak{m} = J' : I$.

Recall that $J \subset J' + \mathfrak{m}I \subset I$. Since $J' : I = \mathfrak{m}$ then $\mathfrak{m}I \subset J'$ and thus $J \subset J' \subset I$. Since J and J' are both minimal reductions of I and $J \subset J'$ then $J = J'$ and thus $J : I = \mathfrak{m}$ as well. $\square$

A careful examination of the above proof shows that it yields even more information about the form a minimal reduction can take. In particular, the coefficients a_i of Corollary 3.4 can be taken to be units.

Corollary 4.5. *Let R and I be as in 4.3, and let J be a minimal reduction of I. Then J is of the form $(e_1 + b_1 e_t, \dots, e_t + b_t e_t, \dots, e_s + b_s e_t)$ for some t, where $b_t = -1$ and for $1 \leqslant i \leqslant s$, either $b_i \notin \mathfrak{m}$ or $b_i = 0$.*

Proof. By Corollary 3.4 there exist $a_i \in R$ such that $J = (e_1 + a_1 e_t, \dots, e_i + a_i e_t, \dots, e_s + a_s e_t)$, where $a_t = -1$. Let $b_i = a_i$ if $a_i \notin \mathfrak{m}$ and $b_i = 0$ if $a_i \in \mathfrak{m}$. Then by the proof of Theorem 4.4 we have that $J = J' = (e_1 + b_1 e_t, \dots, e_i + b_i e_d, \dots, e_s + b_s e_t)$. $\square$

We are now ready to prove the second main theorem of this section.

Theorem 4.6. *Let R and I be as in* 4.3. *Then* $\operatorname{core}(I) = (J : I)I = \mathfrak{m}I$ *for any minimal reduction J of I.*

Proof. By Theorem 4.4 we have $J : I = \mathfrak{m}$ for every minimal reduction J of I. Hence for any minimal reductions J and J' of I we have $J : I = J' : I$. In particular, $(J : I)I \subset J'$ and thus $\mathfrak{m}I = (J : I)I \subset \operatorname{core}(I)$. By Theorem 4.1 we have the other inclusion and thus $\operatorname{core}(I) = (J : I)I = \mathfrak{m}I$. □

Remark 4.7. Let R be a Gorenstein local ring with infinite residue field and I an ideal that satisfies $\operatorname{depth} R/I^j \geqslant \dim R/I - j + 1$ for all $1 \leqslant j \leqslant \ell - g + 1$, where $g = \operatorname{ht} I > 0$. We further assume that I satisfies G_ℓ. This condition is rather mild; it requires that $\mu(I_\mathfrak{p}) \leqslant \dim R_\mathfrak{p}$ for every prime $\mathfrak{p}$ containing I with $\dim R_\mathfrak{p} \leqslant \ell - 1$. Under these assumptions $r(I) \leqslant \ell - g + 1$ is equivalent to $\operatorname{core}(I) = (J : I)J = (J : I)I$ for every minimal reduction J of I as was shown in [2, Theorem 2.6, Corollary 3.7]. Therefore the formula for the core we obtain in Theorem 4.6 is not surprising. We remark that edge ideals of even cycles do satisfy G_ℓ but the depth condition above does not hold for the edge ideals of even cycles of length $d \geqslant 6$ and thus our result does not follow from [2, Theorem 2.6]. Nonetheless the reduction number for these ideals is $r(I) = \frac{d}{2} = \ell - g < \ell - g + 1$ as shown in Lemma 3.5.

Before we can proceed we need to recall some definitions. Let R be a Noetherian ring and I an ideal of $\operatorname{ht} I = g > 0$. For each $i \geqslant g$ a *geometric i-residual intersection* of I is an ideal K such that there exists an i-generated ideal $\mathfrak{a} \subset I$ with $K = \mathfrak{a} : I$, $\operatorname{ht} K \geqslant i$, and $\operatorname{ht}(I + K) \geqslant i + 1$. Furthermore, I is *weakly n-residually S_2* if R/K satisfies Serre's condition S_2 for every geometric i-residual intersection K of I and for all $g \leqslant i \leqslant n$.

The following example shows that the formula for the core given in Theorem 4.6 does not hold in general if I is the edge ideal of a graph with a unique cycle that is even.

Example 4.8. Let G be a graph on the vertices $x_1, \dots, x_6$ with edges $e_1 = x_1x_2$, $e_2 = x_2x_3$, $e_3 = x_3x_4$, $e_4 = x_1x_4$, $e_5 = x_4x_5$, $e_6 = x_5x_6$. Let I be the edge ideal of G in $R = \mathbb{Q}[x_1, \dots, x_6]_{(x_1,\dots,x_6)}$ and let $\mathfrak{m} = (x_1, \dots, x_6)$ denote the maximal ideal of R. Then $\mathfrak{m}I \not\subset \operatorname{core}(I)$. Furthermore, I is not weakly $(\ell - 1)$-residually S_2 and $\operatorname{core}(I)$ is not a finite intersection of general minimal reductions of I.

Proof. Notice that the graph G is a square with two additional edges. By Remark 2.1 we know that $\ell = 5$. Also $g = \operatorname{ht} I = 3$. Let $H = (e_1 + e_2, e_3 + e_2, e_4 + e_2, e_5, e_6 + e_2)$. It is straightforward to verify that $I^2 = HI$ and thus H is a minimal reduction of I. Using Macaulay2 [6] we see that $H : I = (x_1, \dots, x_5)$. Therefore, if $\mathfrak{m}I \subset \operatorname{core}(I)$ then $\mathfrak{m}I \subset H$ and thus $\mathfrak{m} \subset H : I$, a contradiction. Hence $\mathfrak{m}I \not\subset \operatorname{core}(I)$.

We will now show that $\operatorname{core}(I)$ is not a finite intersection of general minimal reductions of I. We follow the outline of the proof of Theorem 4.4. Let ϕ be a presentation matrix of I. Then the matrix ψ of the linear relations on the generators of I is given by

$$\psi = \begin{pmatrix} x_4 & -x_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_1 & -x_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & -x_1 & 0 & x_5 & 0 \\ -x_2 & 0 & 0 & x_3 & -x_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1 & -x_3 & x_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_4 \end{pmatrix}.$$

Let J be a minimal reduction of I. Then by Corollary 3.4 we obtain that $J = (e_1 + a_1e_t, \dots, e_6 + a_6e_t)$, where $1 \leqslant t \leqslant 6$, $a_t = -1$, and $a_j \in R$ for all $1 \leqslant j \leqslant 6$. Let $f_j = e_j + b_je_t$, where $b_j = a_j$ if $a_j \notin \mathfrak{m}$ and $b_j = 0$ if $a_j \in \mathfrak{m}$ for $1 \leqslant j \leqslant 6$. Let $J' = (f_1, \dots, f_6)$. Notice that $f_t = 0$ since $b_t = -1$, and $J \subset J' + \mathfrak{m}I \subset I$. Therefore J' is also a reduction of I by Lemma 3.1. Then $I = (J, e_t)$. We choose elementary row operations so that ϕ' is the new presentation matrix of I that reflects the generating set (J, e_t) of I and ψ' is the corresponding matrix of linear relations. Notice that by the choice of the generating set for I, the t-th row of I forms a (not necessarily minimal) presentation matrix $\tilde{\psi}$ of I/J'. Then $I_1(\tilde{\psi}) = (b_4x_2 - b_1x_4, b_1x_3 - b_2x_1, b_2x_4 - b_3x_2, b_3x_1 - b_4x_3, b_4x_5 - b_5x_1, b_5x_3 - b_3x_5, b_6x_4 - b_5x_6)$ and

$$\tilde{\psi}^T = \begin{pmatrix} 0 & b_4 & 0 & -b_1 & 0 & 0 \\ -b_2 & 0 & b_1 & 0 & 0 & 0 \\ 0 & -b_3 & 0 & b_2 & 0 & 0 \\ b_3 & 0 & -b_4 & 0 & 0 & 0 \\ -b_5 & 0 & 0 & 0 & b_4 & 0 \\ 0 & 0 & b_5 & 0 & -b_3 & 0 \\ 0 & 0 & 0 & b_6 & 0 & -b_5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = B\underline{x}^T.$$

One can show that $I_6(B) = b_5(b_1b_3 - b_2b_4)^2(b_3, b_4, b_5)$. In particular, if $b_5 = 0$ then $I_6(B) = 0$ and thus no maximal submatrix of B is invertible.

Notice that $b_1b_3 \neq b_2b_4$ by Corollary 3.8. Therefore b_3 and b_4 cannot be simultaneously zero. Thus when $b_5 \neq 0$ then $I_6(B) \neq 0$ and therefore B has an invertible 6×6 submatrix and $J' : I = \mathfrak{m}$. Hence $J \subset J' + \mathfrak{m}I = J'$ and thus $J = J'$ and $J : I = \mathfrak{m}$.

Suppose that J is a general minimal reduction of I, i.e. J is generated by ℓ general elements of I. Then $a_j \in \mathbb{Q}$ and thus $b_j = a_j$ for all $1 \leqslant j \leqslant 6$. When J is a general minimal reduction we may choose $b_5 \neq 0$ and thus $J : I = \mathfrak{m}$ for all such J. Hence $\mathfrak{m}I \subset J$ for all general minimal reductions J of I. Therefore $\mathfrak{m}I \subset \bigcap_{J \in \mathcal{M}(I)} J$, where $\mathcal{M}(I) = \{J \mid J$ general minimal reduction of $I\}$. But we already saw that $\mathfrak{m}I \not\subset \operatorname{core}(I)$ and therefore $\operatorname{core}(I) \neq \bigcap_{J \in \mathcal{M}(I)} J$.

Finally, it is straightforward to see that I satisfies G_ℓ. If I were weakly $(\ell-1)$-residually S_2 then by [2, Theorem 4.5] the core would have been a finite intersection of general minimal reductions, a contradiction. Thus I is not weakly $(\ell-1)$-residually S_2. Note that this can also be verified directly. $\square$

Remark 4.9. Notice that Example 4.8 establishes that the condition that I is weakly $(\ell-1)$-residually S_2 is necessary in [2, Theorem 4.5].

5. The core as a finite intersection

We conclude this article by revisiting the question of whether the core may be obtained as a finite intersection of minimal reductions. Recall that under suitable assumptions Corso, Polini and Ulrich prove that the core may be obtained as a finite intersection of general minimal reductions [2, Theorem 4.5]. Note that Example 4.8 is an instance where the assumptions of [2, Theorem 4.5] fail to hold and the core is not a (finite) intersection of general minimal reductions. We will prove in this section that when I is the edge ideal of an even cycle, then $\operatorname{core}(I)$ is obtained as a finite intersection of minimal reductions and we will give an explicit description of these minimal reductions. We first show that the edge ideal corresponding to an octagon is not weakly $(\ell-1)$-residually S_2.

Example 5.1. Let I be the edge ideal of an even cycle of length 8. Let R be the corresponding localized polynomial ring over $\mathbb{Q}$. Then I is not weakly $(\ell-1)$-residually S_2.

Proof. Let $I = (e_1, \dots, e_8)$. Then $\ell = 7$. Let $\mathfrak{a} = (e_1 + e_7 - e_8, e_2 + e_7 + 3e_8, e_3 + e_7 + e_8, e_4 + e_7 + e_8, e_5 + e_7 + e_8, e_6 + e_7 + 2e_8)$ and $K = \mathfrak{a} : I$. Then $\operatorname{ht} K = 6$ and $\operatorname{ht}(I + K) = 7$. Therefore K is a geometric 6-residual intersection of I. Using Macaulay2 [6] we have that $\operatorname{projdim}(R/K) = 7$ and thus $\operatorname{depth} R/K = 1$, which then means R/K does not satisfy Serre's condition S_2. $\square$

When I is the edge ideal of an even cycle then I need not be weakly $(\ell-1)$-residually S_2 as Example 5.1 suggests. Thus we may not apply [2, Theorem 4.5]. Instead, we will employ different methods.

Notation 5.2. Let $I = (e_1, \dots, e_d)$ be the edge ideal of an even cycle. For every $1 \leqslant t \leqslant d$, let $J_t = (e_1 + a_1e_t, e_2 + a_2e_t, \dots, e_d + a_de_t)$, where $a_i = 1$ for $i \neq t$ and $a_t = -1$. For every $1 \leqslant t \leqslant d/2$ we define the following ideals:

$L_{2t} = (e_1 + a_1e_{2t}, \dots, e_d + a_de_{2t})$, where $a_i = 1$ for all $i \neq 2t$ even, $a_i = 0$ for all i odd, and $a_{2t} = -1$;

$H_{2t} = (e_1 + a_1e_{2t}, \dots, e_d + a_de_{2t})$, where $a_i = 1$ for all i odd, $a_i = 0$ for all $i \neq 2t$ even, and $a_{2t} = -1$;

$H_{2t-1} = (e_1 + a_1e_{2t-1}, \dots, e_d + a_de_{2t-1})$, where $a_i = 1$ for all i even, $a_i = 0$ for all $i \neq 2t-1$ odd, and $a_{2t-1} = -1$.

Remark 5.3. Let I be the edge ideal of an even cycle $e_1, \dots, e_d$. Using the same techniques as in Examples 3.10 and 3.11, we see that for every $1 \leqslant t \leqslant d/2$, the ideals L_{2t}, H_{2t}, H_{2t-1} in 5.2 are minimal reductions of I. When $\operatorname{char} k \neq 2$ then J_t is a minimal reduction of I for every $1 \leqslant t \leqslant d$, by Example 3.9.

Proposition 5.4. *Let I be the edge ideal of an even cycle $e_1, \dots, e_d$. Let $d = 2n$ for some integer $n \geqslant 2$. Let $p = \operatorname{char} k \geqslant 0$. If $p \neq 2$ and $n \not\equiv 1 \bmod p$ then* $\operatorname{core}(I) = \bigcap_{t=1}^{d} J_t$.

Proof. First recall that $\operatorname{core}(I) = \mathfrak{m}I$, by Theorem 4.6. Let $C = \bigcap_{t=1}^{d} J_t$. Since J_t is a minimal reduction of I for each t, we have that $\mathfrak{m}I \subset C$.

In order to establish the other inclusion, suppose $f \in C \setminus \mathfrak{m}I$. Since $f \in I$ then we may write $f = \sum_{i=1}^{d} h_ie_i$, for some $h_i \in R$. Now since (by clearing denominators in the localization if necessary) h_i can be taken to be a polynomial then we may write $h_i = g_i + h'_i$, for some $g_i \in \mathfrak{m}$ and $h'_i \in k$ of degree 0. Notice that $g_ie_i \in \mathfrak{m}I \subset J_t$ for all i, t. Thus if $g = \sum_{i=1}^{d} g_ie_i$, then $g \in \mathfrak{m}I \subset C$ and so $f' = f - g = \sum_{i=1}^{d} h'_ie_i \in C \setminus \mathfrak{m}I$. Therefore, without loss of generality, we may assume $f = \sum_{i=1}^{d} h_ie_i$, where $h_1 = 1$ and $h_i \in k$ for all i.

We observe that since $f \in J_1$ then we may write $f = \sum_{i=2}^{d} a_i(e_i + e_1)$, for some $a_i \in R$. Notice that f is homogeneous of degree 2 and thus we may assume $a_i \in k$ since all terms of higher degree must cancel. The set $\{e_1, \dots, e_d\}$ is linearly independent over k. Therefore we may equate coefficients of e_i in the two summation representations of f. Thus $h_i = a_i$ for $i \geqslant 2$. Furthermore, by equating the coefficients of e_1 we have $\sum_{i=2}^{d} h_i = 1$. Since $f \in J_2$, then $f = b_1(e_1 + e_2) + \sum_{i=3}^{d} b_i(e_i + e_2)$, for some $b_i \in k$. Using the same method as above we obtain $b_i = h_i$ for all $i \neq 2$. By examining e_2, and recalling that $h_1 = 1$, we see that $1 + \sum_{i=3}^{d} h_i = h_2$ and thus $1 + \sum_{i=2}^{d} h_i = 2h_2$. Combining both equations yields $2 = 2h_2$. Since $p \neq 2$, we have $h_2 = 1$ and $\sum_{i=3}^{d} h_i = 0$.

We will proceed by induction. Suppose that for some $t < d$, $h_i = 1$ for all $i \leqslant t$ and $\sum_{i=t+1}^{d} h_i \equiv 2 - t$, where equivalence will be considered modulo p. Since $f \in J_{t+1}$ then $f = \sum_{i=1}^{t}(e_i + e_{t+1}) + \sum_{i=t+2}^{d} h_i(e_i + e_{t+1})$. Examining the coefficient of e_{t+1} yields $h_{t+1} \equiv t + \sum_{i=t+2}^{d} h_i$ and therefore $2h_{t+1} \equiv t + \sum_{i=t+1}^{d} h_i$, or $2h_{t+1} \equiv t + 2 - t$. So $h_{t+1} \equiv 1$ and thus $\sum_{i=t+2}^{d} h_i \equiv 1 - t = 2 - (t+1)$. Thus by induction, we may assume $h_i \equiv 1$ for all $i \leqslant d-1$ and $\sum_{i=t+1}^{d} h_i \equiv 2 - t$ for all $t \leqslant d-1$. Note that since $h_i \in k$, $h_i \equiv 1$ implies $h_i = 1$ in k. Now assume $t = d-1$. Then $h_d = \sum_{i=d}^{d} h_i \equiv 2 - (d-1) = 3 - d$. Again, since $f \in J_d$ then $f = \sum_{i=1}^{d-1}(e_i + e_d)$ and thus $h_d = d - 1$. But $h_d \equiv 3 - d$, so $d - 1 \equiv 3 - d$ or $d \equiv 2$. Equivalently, since $d = 2n$ then $n \equiv 1$, which is a contradiction. Therefore $C \subset \mathfrak{m}I$. $\square$

We now consider the remaining cases when the characteristic of the residue field is 2 or $n \equiv 1 \bmod p$.

Proposition 5.5. *Let I be the edge ideal of an even cycle $e_1, \dots, e_d$. Let $d = 2n$ for some integer $n \geqslant 2$. Let $p = \operatorname{char} k \geqslant 0$. If $p = 2$ or $n \equiv 1 \bmod p$ then*

$$\operatorname{core}(I) = \bigcap_{i=1}^{d} H_i \cap \bigcap_{t=1}^{d/2} L_{2t},$$

where H_i and L_{2t} are as in 5.2.

Proof. Let $C = \bigcap_{i=1}^{d} H_i \cap \bigcap_{t=1}^{d/2} L_{2t}$. By Theorem 4.6, $\operatorname{core}(I) = \mathfrak{m}I$. Since for every $1 \leqslant t \leqslant d/2$ and every $1 \leqslant i \leqslant d$ we have that L_{2t} and H_i are all minimal reductions of I, then $\mathfrak{m}I \subset C$. As before, we may assume $f \in C \setminus \mathfrak{m}I$ and $f = \sum_{i=1}^{d} h_ie_i$, where $h_1 = 1$ and $h_i \in k$ for all i.

First we note that since $f \in H_1$ we may write $f = \sum_{i=1}^{d/2} a_{2i}(e_{2i} + e_1) + \sum_{i=2}^{d/2} a_{2i-1}e_{2i-1}$ for some $a_i \in k$. Equating coefficients yields $a_i = h_i$ for all $i \neq 1$ and that $\sum_{i=1}^{d/2} a_{2i} = \sum_{i=1}^{d/2} h_{2i} = 1$.

Since $f \in L_d$ then $f = \sum_{i=1}^{d/2-1} b_{2i}(e_{2i} + e_d) + \sum_{i=1}^{d/2} b_{2i-1}e_{2i-1}$ for some $b_i \in k$. Equating coefficients as before, we have that $\sum_{i=1}^{d/2-1} h_{2i} = h_d$ and thus $\sum_{i=1}^{d/2} h_{2i} = 2h_d$. Hence $2h_d = 1$. If $\operatorname{char} k = 2$ then we have that $0 = 1$, which is a contradiction. Thus we may assume that $\operatorname{char} k \neq 2$, $n \equiv 1 \bmod p$ and $2h_d = 1$.

Similarly, since $f \in L_{d-2}$ we obtain $\sum_{i=1}^{d/2-2} h_{2i} + h_d = h_{d-2}$ and hence $\sum_{i=1}^{d/2} h_{2i} = 2h_{d-2}$. Thus $2h_{d-2} = 1$. Repeating this process yields $2h_{2i} = 1$ for all $1 \leqslant i \leqslant d/2$. But as $\sum_{i=1}^{d/2} h_{2i} = 1$ we have $2\sum_{i=1}^{d/2} h_{2i} = \frac{d}{2} = 2$, i.e. $d \equiv 4 \bmod p$. Since $d = 2n$ then $n \equiv 2 \bmod p$, which is a contradiction. Thus $C \subset \mathfrak{m}I$. $\square$

Theorem 5.6. *Let I be the edge ideal of an even cycle. Then* $\operatorname{core}(I)$ *is obtained as a finite intersection of minimal reductions of I.*

Proof. Combine Propositions 5.4 and 5.5. $\square$

Remark 5.7. Let I be the edge ideal of an even cycle. Recall that the $\operatorname{gradedcore}(I)$ is the intersection of all homogeneous minimal reductions of I. In general, $\operatorname{core}(I) \subset \operatorname{gradedcore}(I)$. We note that all the reductions in 5.2 are homogeneous minimal reductions. Hence $\operatorname{gradedcore}(I) \subset C$, where C is as in Propositions 5.4 and 5.5. Therefore, $\operatorname{core}(I) = \operatorname{gradedcore}(I) = \mathfrak{m}I$, by Theorem 5.6.

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